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Fixed Point Limit Behavior of N-Mode Truncated Navier–Stokes Equations as N Increases

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The fixed point behavior of N-mode truncations of the Navier–Stokes equations on a two-dimensional torus is investigated as N increases. From $N = 44$ on the behavior does not undergo any qualitative change. Furthermore, the bifurcations occur at critical parameter values which clearly tend to stabilize as N approaches 100.

KEY WORDS: Fixed point; limit behavior; stream function; bifurcation theory; truncated Navier–Stokes equations.

1. INTRODUCTION

The advent of modern computers has made possible the numerical investigation of nonlinear systems of ordinary differential equations derived through truncations of suitable expansions of partial differential equations governing fluid flows. Two reasons motivate the study of truncated models. First, an approach is attempted to the solution of equations not explicitly solvable. The hope is that, if the number of modes used in the truncation is sufficiently large, the solution will exhibit a qualitative behavior not too unlike the true one. Second, truncated models can be very interesting as dynamical systems which undergo significant changes in behavior when an external parameter assumes some critical values. The transitions which lead to a chaotic regime are of particular interest.

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The first work in this context, published in 1963, is a well-known paper by Lorenz[1] which concerns a 3-mode truncation of the equations for convection between parallel plates. Fifteen years later Curry[2] considered a 15-mode extension of the Lorenz System. Yahata, in a series of papers (see, for example, Ref. 3), studied the Taylor vortices of the Couette flow after truncation of the Navier–Stokes equations with the aid of the Galerkin method. Orszag developed sophisticated techniques, mainly based on spectral methods, for a different numerical solution of the Navier–Stokes equations. These techniques were used to simulate different fluid systems with a very large number of degrees of freedom (see Refs. 4 and 5 and references therein). Together with collaborators, we investigated several systems derived from the Navier–Stokes equations on a two-dimensional torus,[6–11] from a minimal 4-mode truncation to a largest 18-mode one. Such numerical work has contributed in a substantial way to the present knowledge of chaotic phenomena in dynamical systems.

This paper is concerned with the attempt of studying the planar Navier–Stokes equations with periodic boundary conditions through truncations. Our previous studies have shown that the behavior of truncated models is highly sensitive to the choice of the modes used in the truncation. If the number of modes is small, the addition or substitution of only one mode can radically change the phenomenology of the model. This fact leads us to conclude that only considering a large number of modes can we hope to find a limit behavior, that is, a behavior which is not affected by addition or change of modes. The question is whether such a purpose can be achieved with a number of modes which still allows actual investigation by a computer. Interesting theoretical results, providing an estimate to the number of modes sufficient to obtain correct approximate solutions of the two-dimensional Navier–Stokes equations, have been recently obtained by Foias *et al.*[12] These results, however, do not appear useful to answer our question.

Here we confine ourselves to investigating only the behavior of the fixed points. By considering the N-mode truncation associated with all the modes included in a ball, we obtain a system of differential equations which becomes larger and larger as the radius of the ball is increased. From $n = 44$ on, the phenomenology concerning the fixed points does not undergo any qualitative change. In addition, the critical values of the parameter corresponding to the bifurcations clearly tend to stabilize as N varies up to $N = 98$. It is then an interesting example of N-mode truncated Navier–Stokes equations which show a qualitative and quantitative limit behavior as the number N of modes increases.

2. THE TRUNCATED EQUATIONS

Consider the Navier–Stokes equations for an incompressible fluid on the torus $T^2 = [0, 2\pi] \times [0, 2\pi]$:

$$\begin{aligned} &\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla p + \mathbf{f} + \nu \Delta \mathbf{u} \\ &\operatorname{div} \mathbf{u} = 0 \\ &\int_{T^2} \mathbf{u}\, d\mathbf{x} = 0 \\ &\mathbf{u}|_{t=0} = \mathbf{u}_0 \end{aligned} \tag{1}$$

where $\mathbf{u}$ is the velocity field, p is the pressure, ν is the viscosity, and $\mathbf{f}$ a periodic volume force. Expand $\mathbf{u}$, $\mathbf{f}$, and p in Fourier series:

$$\mathbf{u}(\mathbf{x}, t) = \sum_{\mathbf{k} \neq 0} \gamma_{\mathbf{k}}(t) e^{i\mathbf{k} \cdot \mathbf{x}} \frac{\mathbf{k}^{\perp}}{|\mathbf{k}|} \tag{2}$$

$$\mathbf{f}(\mathbf{x}, t) = \sum_{\mathbf{k} \neq 0} \left[f_{\mathbf{k}}(t) \frac{\mathbf{k}^{\perp}}{|\mathbf{k}|} + \tilde{f}_{\mathbf{k}}(t) \frac{\mathbf{k}}{|\mathbf{k}|} \right] e^{i\mathbf{k} \cdot \mathbf{x}} \tag{3}$$

$$p(\mathbf{x}, t) = \sum_{\mathbf{k} \neq 0} p_{\mathbf{k}}(t) e^{i\mathbf{k} \cdot \mathbf{x}} \tag{4}$$

$\mathbf{k} = (k_x, k_y)$ being a wave vector with integer components ("mode") and $\mathbf{k}^{\perp} = (k_y, -k_x)$. By imposing the reality condition $\gamma_{-\mathbf{k}} = -\bar{\gamma}_{\mathbf{k}}$, Eqs. (1) yield the following ordinary differential equations for $\{\gamma_{\mathbf{k}}(t)\}_{\mathbf{k} \neq 0}$:

$$\dot{\gamma}_{\mathbf{k}} = -\nu |\mathbf{k}|^2 \gamma_{\mathbf{k}} - i \sum_{\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k} = 0} \frac{(\mathbf{k}_1^{\perp} \cdot \mathbf{k}_2)(k_2^2 - k_1^2)}{2|\mathbf{k}_1|\,|\mathbf{k}_2|\,|\mathbf{k}|} \bar{\gamma}_{\mathbf{k}_1} \bar{\gamma}_{\mathbf{k}_2} + f_{\mathbf{k}} \tag{5}$$

Once the $\gamma_{\mathbf{k}}$'s are obtained, the $p_{\mathbf{k}}$'s are given by

$$p_{\mathbf{k}} = -\sum_{\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k} = 0} \frac{(\mathbf{k}_1^{\perp} \cdot \mathbf{k}_2)(\mathbf{k}_1 \cdot \mathbf{k}_2^{\perp})}{|\mathbf{k}_1|\,|\mathbf{k}_2|\,|\mathbf{k}|^2} \bar{\gamma}_{\mathbf{k}_1} \bar{\gamma}_{\mathbf{k}_2} + \tilde{f}_{\mathbf{k}} \tag{6}$$

If L is a finite set of $2N$ wave vectors such that if $\mathbf{k} \in L$, also $-\mathbf{k} \in L$, we define the N-mode truncated Navier–Stokes equations as

$$\begin{aligned} &\dot{\gamma}_{\mathbf{k}} = -\nu |\mathbf{k}|^2 \gamma_{\mathbf{k}} - i \sum_{\substack{\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k} = 0 \\ \mathbf{k}_1, \mathbf{k}_2 \in L}} \frac{(\mathbf{k}_1^{\perp} \cdot \mathbf{k}_2)(k_2^2 - k_1^2)}{2|\mathbf{k}_1|\,|\mathbf{k}_2|\,|\mathbf{k}|} \bar{\gamma}_{\mathbf{k}_1} \bar{\gamma}_{\mathbf{k}_2} + f_{\mathbf{k}}, \qquad k \in L \\ &\gamma_{-\mathbf{k}} = -\bar{\gamma}_{\mathbf{k}} \end{aligned} \tag{7}$$

This system consists of N differential equations in the complex unknowns $\gamma_{\mathbf{k}}(t)$, that is $2N$ equations in real variables. In the following we will often refer to L as if constituted only by the N vectors placed in the half-space

$$\Pi^+ = \{(x, y), x \geqslant 0, y \geqslant 0, \text{ if } x = 0\}$$

Now, for the sake of simplicity, we suppose that the force $\mathbf{f}$ acts only on some mode k^* and is independent of time. As has been shown in Ref. 13, under these hypotheses $f_{\mathbf{k}^*}$ can be taken real without losing generality. As a consequence system (7) admits particular solutions in which each $\gamma_{\mathbf{k}}(t)$ is either real or imaginary pure. The consideration of such a particular solution leads to the study of a system of N rather than $2N$ equations.

Let $L^{(M)}$ be the set of modes $\mathbf{k}$ such that $|\mathbf{k}|^2 = k_x^2 + k_y^2 \leqslant M$, with M sum of two squared integers. As M is increased from $M = 8$, the truncations associated with $L^{(M)}$ provide a sequence of nontrivial models, each of them representing an enlargement of all the previous ones. To make possible the comparison with the truncations already studied,[6-11] we assume that $\mathbf{k}^* = (2, -1)$ is the only mode excited by the force $\mathbf{f}$, with $f_{\mathbf{k}^*} = R$. Letting $\nu = 1$ by a rescaling of length and time, the external parameter R can be referred to as the Reynolds number because of its physical meaning.

Under the above assumptions there exist infinitely many N-dimensional hyperplanes, subspaces of the $2N$-dimensional phase space, which are invariant with respect to the flow defined by system (7). These hyperplanes are symmetrically placed because of a one-parameter group of angular symmetries (see again Ref. 13). Assuming $\gamma_{\mathbf{k}}(t) = \rho_{\mathbf{k}}(t)e^{i\theta_{\mathbf{k}}(t)}$, with both $\rho_{\mathbf{k}}(t)$ and $\theta_{\mathbf{k}}(t)$ varying in $(-\infty, +\infty)$, if α and β are real parameters, $\alpha \in [0, 2\pi)$, $\beta \in [0, \pi)$,

$$S_\alpha : \{\theta_{\mathbf{k}}(t) \to \theta_{\mathbf{k}}(t) + (k_x + 2k_y)\alpha\}_{\mathbf{k} \in L^{(M)}} \tag{8}$$

defines a one-parameter group of symmetries and

$$H_\beta : \left\{\theta_{\mathbf{k}} = (k_x + 2k_y)\beta + (1 - k_x - k_y)\frac{\pi}{2}\right\}_{\mathbf{k} \in L^{(M)}} \tag{9}$$

represents a linear continuum of stationary solutions for $\{\theta_{\mathbf{k}}(t)\}_{\mathbf{k} \in L^{(M)}}$.

The particular solution we consider here corresponds to the hyperplane $H_{\pi/2}$ (an equivalent one would be H_0). For $\beta = \pi/2$, in fact, each $\theta_{\mathbf{k}}$ in (9) is an integer multiple of $\pi/2$, which means that each $\gamma_{\mathbf{k}}$ is either real or purely imaginary. To study such a solution is the same as restricting the choice of the initial data for (7) to the points of $H_{\pi/2}$. In Ref. 13 support is given to the fact that, at least for Reynolds numbers below some critical threshold, the particular solution displays the whole behavior of the complex equations.

The adoption of a particular solution permits us to deal with a set $\{x_{\mathbf{k}}(t)\}_{\mathbf{k}\in L^{(M)}}$ of real variables. There exists a transformation

$$T:\left\{x_{\mathbf{k}}\to(-1)^{k_x}x_{\mathbf{k}}\right\}_{\mathbf{k}\in L^{(M)}}$$

which leaves unchanged the systems of equations we study. Besides the symmetry T, which is valid in any case, another symmetry τ works when all the variables $x_{\mathbf{k}}$ relative to $\mathbf{k}$'s with odd k_x are always null. The partial symmetry is defined in the following way:

$$\tau:\left\{x_{\mathbf{k}}\to(-1)^{k_x/2+k_y}x_{\mathbf{k}}\right\}_{\mathbf{k}\in L^{(M)}}$$

The knowledge of T and τ is essential to the understanding of the bifurcation diagram which describes the fixed point behavior we are interested in.

The largest truncation we take into account is that corresponding to $L^{(64)}$, which includes the 98 lowest modes of Π^+. So, as M varies between 8 and 64, we are concerned with a sequence of systems of first-order, nonlinear, ordinary differential equations with a maximum configuration of 98 equations and more than 5200 nonlinear terms. The 98-mode system, which cannot be given here for obvious reasons, was written, together with its Jacobian, by the computer directly in the form of FORTRAN subroutines. These routines were arranged in such a way to allow the user also to select, among all the 98 modes, the ones associated with any value of M less than 64.

3. FIXED POINT BEHAVIOR

The behavior of the fixed points in the systems $S^{(M)}$ is quite complicated for small M. As M increases, it tends to become simpler remaining unchanged for $M \geqslant 26$. Two different sets of fixed points are present: to distinguish one from the other they will be denoted with the two letters P and Q.

The trivial point P_0, having all the components zero except that associated with the mode excited by the forcing, is at the origin of the first set. P_0, which exists for all the values of the Reynolds number R, becomes unstable at a critical value R_1. For all M but $M=8$ the instability of P_0 is due to a real eigenvalue of the Jacobian of $S^{(M)}$ becoming positive. So, two new symmetric fixed points P_γ, $\gamma=\pm$, bifurcate from P_0. They are changed into each other by the symmetry T and have coordinates zero except those relative to $\mathbf{k}$ with the difference (k_y-k_x) multiple of three. For $M \leqslant 13$, two other pairs of neighboring fixed points $(\bar{P}_\gamma, \bar{P}^*_\gamma)$, arisen via a tangent bifurcation, can interfere with the P_γ's originating involved phenomenologies that we do not describe. For larger M the $\bar{P}_\gamma$'s and the $\bar{P}_\gamma^*$'s no longer appear and the P_γ's behave in a simple way: they remain stable up to a critical value R_5, when a pair of complex conjugate eigenvalues of the Jacobian crosses the imaginary axis and a Hopf bifurcation occurs.

Consider now the parallel history of the fixed points Q. It originates at R_2 when two pairs of fixed points (Q_σ, Q_δ^*), $\delta = \pm$, Q_δ stable and Q_δ^* unstable, appear via a tangent bifurcation. The points Q_δ and Q_δ^* have coordinates $x_{\mathbf{k}}$ which are zero in correspondence of the $\mathbf{k}$'s with odd k_x. As a consequence the partial symmetry τ holds and the Q_δ's, as well as the Q_δ^*'s, are mutual images under τ of each other, which yet does not imply identical behavior. Then we have to follow the two distinct evolutions of the stable Q_δ's (the Q_δ^*'s, although they produce interesting bifurcation phenomena, remain always unstable).

As was true for the P_γ's, for small M the behavior of the Q_δ's, sensibly dependent on M, can be very intricate. Also on this case, in order not to make too heavy the description, let us devote our attention only to the limit behavior which comes out from $M = 26$ on. The fixed point Q_+ is stable up to $R = R_2$, when a Hopf bifurcation takes place. On the other hand, the

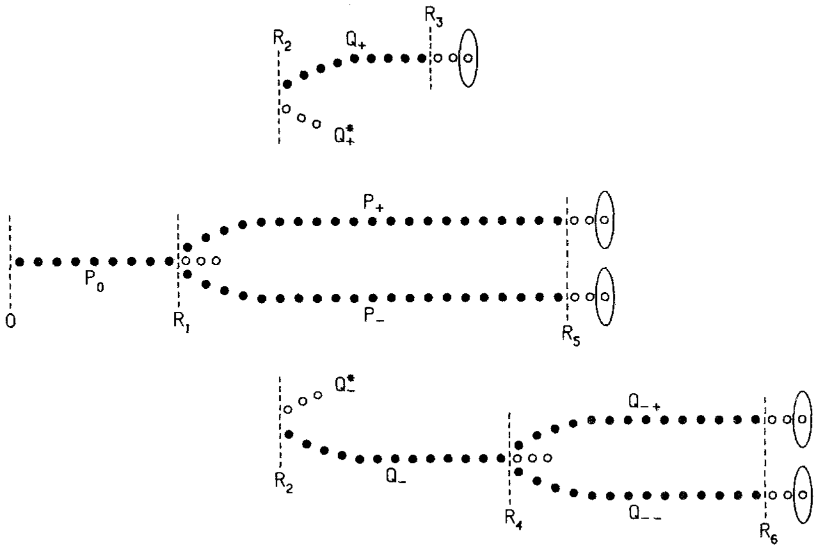

Fig. 1. Bifurcation diagram of the fixed points for N-mode truncations with $N \geqslant 44$. Solid circles represent stable fixed points, open circles unstables ones, ellipses attracting periodic orbits.

point Q_- becomes unstable at R_4 bifurcating into the stable $Q_{-\epsilon}$'s, $\epsilon = \pm$, which are transformed into each other by the symmetry T. At R_6 also the $Q_{-\epsilon}$'s lose stability via Hopf bifurcation.

The qualitative behavior common to all the models $S^{(M)}$ with $M \geqslant 26$ is represented in Fig. 1. The ellipses there indicate that attracting periodic orbits arise from the points Q_+, P_γ, and $Q_{-\epsilon}$ when they become unstable. We verified this fact by integrating the system $S^{(26)}$ for Reynolds number R slightly larger than R_3, R_5, R_6, with initial conditions close to Q_+, P_γ, and $Q_{-\epsilon}$, respectively. These were the only cases in which we integrated the differential equations during our numerical investigation.

Interesting considerations can be made also if we consider the results of our investigation from a qualitative point of view. Table I collects the numerical values of the critical bifurcation points R_i, $i = 1, \ldots, 6$, for M increasing from 26 to 64, that is for a number of modes N varying from 44 up to 98. The table clearly shows that, as M increases, each R_i is subject to small changes, completely negligible in the cases of R_1 and R_2. It appears evident that all the R_i's tend to stabilize.

Because each set $L^{(M)}$ consists of all the modes $\mathbf{k}$ included in a ball of radius $\sqrt{M}$, the truncations (7) with $L = L^{(M)}$ represent a natural way of approximating the infinite system (5). So, even if in a context restricted by some simplifying hypotheses, it seems well justified to talk about a qualitative and quantitative limit behavior of the fixed points of Eqs. (1).

One can ask whether this limit behavior is maintained or not when we consider sequences of truncations associated with sets L which become

Table I. Critical Parameter Values R_i, $i = 1, \ldots, 6$, for the N-Mode Truncations $S^{(M)}$, from $M = 26$ to $M = 64$

M	N	R_1	R_2	R_3	R_4	R_5	R_6
26	44	19.78	31.71	50.01	57.18	64.28	87.79
29	48	19.78	31.70	49.04	56.46	67.69	90.23
34	54	19.78	31.50	49.38	57.08	67.36	89.28
36	56	19.78	31.50	49.35	57.14	68.11	89.10
37	60	19.78	31.53	49.23	57.23	65.49	88.21
40	64	19.78	31.55	49.30	57.20	64.85	86.71
41	68	19.78	31.56	49.28	57.70	65.00	88.95
45	72	19.78	31.58	49.24	58.61	64.53	88.61
49	74	19.78	31.58	49.24	58.62	64.73	88.69
50	80	19.78	31.58	49.30	58.86	65.31	87.37
52	84	19.78	31.58	49.45	58.48	65.33	87.92
53	88	19.77	31.58	49.46	58.46	65.33	88.21
58	92	19.77	31.58	49.47	58.61	65.34	87.98
61	96	19.77	31.58	49.45	58.62	65.34	87.94
64	98	19.77	31.58	49.45	58.62	65.37	87.96

larger and larger excluding some vectors **k** in some systematic way keeping the symmetries T and τ. A few experiments, made with the aim of answering such a question, seem to show that the behavior of the fixed points always tends to a limit as the number of modes increases over 50. This limit, however, can change both qualitatively and quantitatively.

To conclude the section, two more remarks appear opportune. First, the present numerical investigation, which basically is founded on bifurcation theory and Newton's method to the solution of an equation $\mathbf{F}(\mathbf{x}) = 0$, $x \in R^N$, is considerably facilitated by the knowledge of the fixed point structure. It is important to stress the fact that such a structure, well known to us in consequence of a complete study of systems, $S^{(8)}$, $S^{(9)}$, and $S^{(10)}$,[10] is strictly connected to the ball form of the sets $L^{(M)}$ and is likely to change with different truncations. Second, we warn that any comparison with previous results must take into account a rescaling factor for the Reynolds number. The present values of R must be divided by a factor $5\sqrt{10}$ to obtain the R scale of Refs. 6 and 11 and by $5\sqrt{2}$ in the remaining cases (Refs. 7–10).

4. DESCRIPTION OF THE FLUID FLOW

It seems interesting at this point to visualize the actual flow of the fluid in the stationary states associated with the fixed points P and Q. Being concerned with a two-dimensional incompressible fluid, the natural description is provided by a stream function $\psi(x, y)$ such that $\mathbf{u} = \mathbf{e}_z \times \nabla\psi$, where $\mathbf{e}_z$ is a unit vector normal to the plane of the fluid.

Figures 2 and 3, relative to the points P and Q, respectively, show the streamfield at different values of the parameter R. While in P_0 the flow is completely aligned with the forcing, vortexlike structures develop in P_γ. As R increases they grow stronger and the flow undergoes substantial changes because, besides $\mathbf{k}^*$, the modes $(1, 1)$ and $(1, -2)$ also acquire relevance, with the former, the lowest excited in P_γ, becoming dominant.

The points Q have a vortex structure already at their appearance, with a number of modes involved. The mode $(0, 1)$, the lowest excited in Q_δ, is always relevant together with $\mathbf{k}^*$ and becomes dominant as R increases. Some other modes, however, remain important and they originate a smaller scale superimposed to the main vortices. This is even more true for the points $Q_{-\epsilon}$, in which all the modes are activated, with $(1, 0)$ also becoming relevant.

5. CONCLUSION

We reported the result of a numerical study about the fixed point behavior of N-mode truncations of the planar Navier–Stokes equations

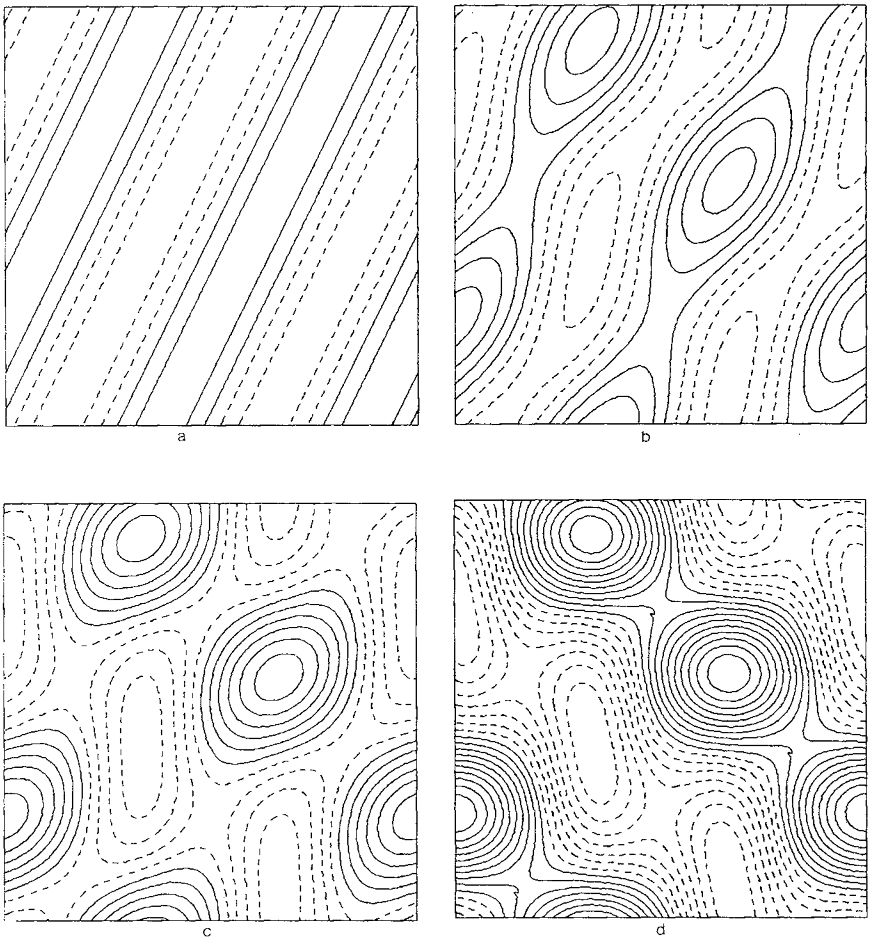

Fig. 2. Streamfield representation for P_0 at (a) $R = 35$ and for P_- at (b) $R = 45$, (c) $R = 65$, (d) $R = 125$. Continuous lines correspond to positive values of the stream function, broken lines to negative ones.

with periodic boundary conditions. Under the simplifying assumption of an external force independent of time and acting on one mode only, truncated equations exhibit a qualitative and quantitative limit behavior as N is increased. When each truncation is obtained by using all and only the model included in a ball, from $N = 44$ the phenomenology does not go through any qualitative change. While this was not unexpected, it may appear surprising that the critical parameter values reach a good stabilization as N tends toward 100.

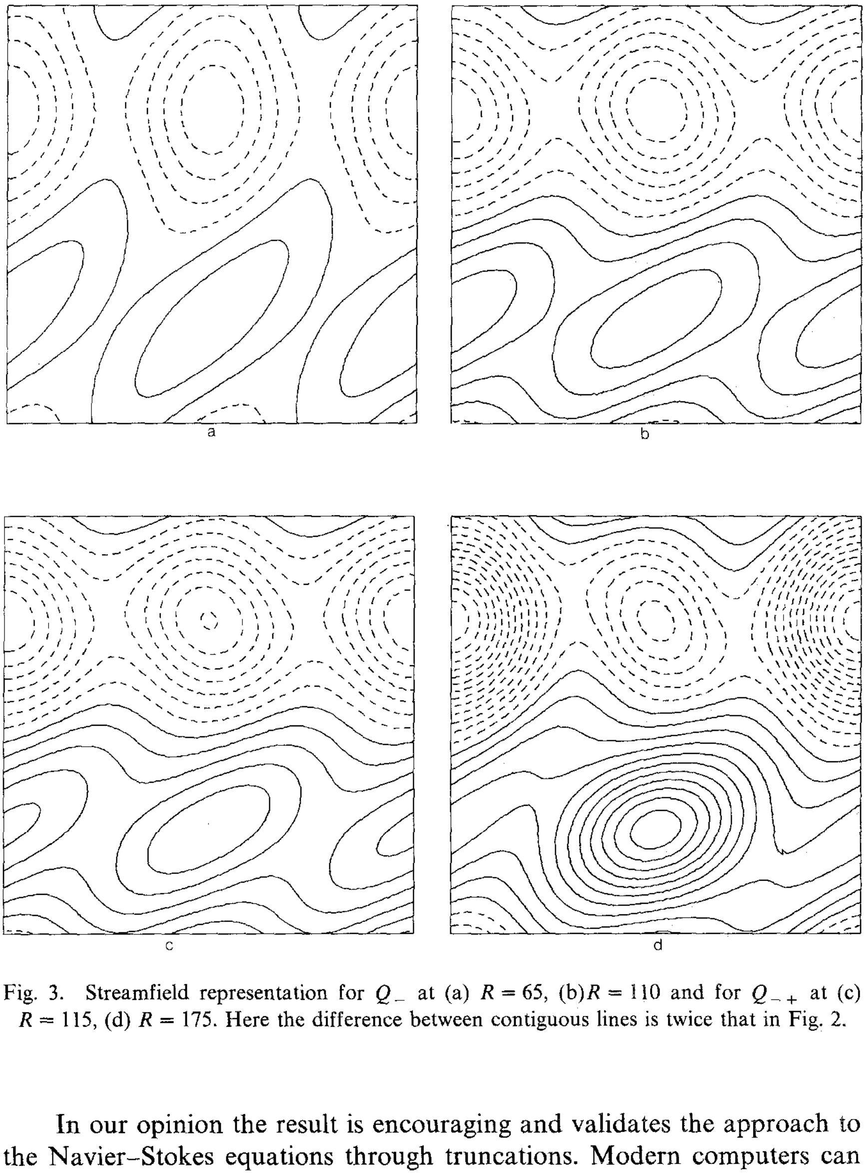

Fig. 3. Streamfield representation for Q_- at (a) $R = 65$, (b)$R = 110$ and for Q_{-+} at (c) $R = 115$, (d) $R = 175$. Here the difference between contiguous lines is twice that in Fig. 2.

In our opinion the result is encouraging and validates the approach to the Navier–Stokes equations through truncations. Modern computers can certainly allow a detailed investigation, based on bifurcation theory, also of periodic and quasiperiodic attractors for systems of even more than one hundred differential equations. This strengthens our confidence in the possibility of finding a limit behavior for most of the nonchaotic attractors, and therefore our confidence in a satisfactory description of the onset of chaos.

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